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A Dynamic Model of a Reinforced Thin Plate with Ribs of Finite Width

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PREFACE

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14. ABSTRACT

This report derives an analytical model of a thin plate of infinite spatial extent stiffened by ribs that have a finite width. The governing equation is a flexural plate model that has an external load and is reinforced by an infinite number of equally spaced ribs. The Heaviside step function is used to load the rib forces onto the plate. A Fourier series then replaces the Heaviside step function and the equation decouples using an orthogonalization procedure. The resulting system can be represented by an infinite set of algebraic equations that are truncated to find a solution to the plate displacement. An example problem is formulated and the results are discussed.

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1. INTRODUCTION

Reinforced plates and shells have a variety of applications. For instance, they are present in the design of ships, undersea vehicles, and aircraft because reinforcement provides increased structural strength with minimal increased weight. While reinforcing a structure will allow it to survive in environments in which an unreinforced structure will fail, the resulting dynamic response of the structure changes dramatically. Early work in the area of reinforced plates[1-6] generally modeled the reinforcement of the plate as a line stiffener using a Dirac delta function to mathematically represent the effects. Mace[1-2] modeled the response of periodically stiffened fluid-loaded plates to harmonic loading and to line and point force loading. Mace's work involved a transformation into the wavenumber domain and the evaluation of a contour integral. Stepanishen[3] modeled the scattering characteristics of a plate with line impedance discontinuities. To evaluate the scattered pressure, he used a Fourier integral to calculate the relationship between the plate velocity in the wavenumber domain and the spatial domain. Eatwell and Butler[4] determined the vibration sound radiation from a fluid-loaded plate stiffened by a finite number of beams. Their solution involves an asymptotic evaluation of the pressure field by two Fourier integrals. Cray[5] determined the response of a sectionally aperiodic plate to a line force in the wavenumber domain. Recently, Hull[6] derived the elastic response of a thick plate system to harmonic loading with stiffeners. Some work exists where the stiffeners are modeled with finite width. Woolley[7,8] modeled the acoustic scattering from a plate reinforced by a single rib and by a finite number of ribs. In these papers, Woolley formulates the problem in the wavenumber domain and then solves it using a complicated method of contour integration. Woolley allows ribs of finite width in his model; however, he states that another author (Stepanishen) has obtained a different result." The specific problem of a spatial domain response

of a thin plate reinforced by ribs of finite width subjected to convective loading has not yet been addressed.

This report derives an analytical model of a thin plate of infinite spatial extent stiffened by ribs that have a finite width. The model developed here is different from previous models as it is based on differential equation theory in the spatial domain only and, thus, does not involve a complicated integral to be evaluated. The governing equation is a flexural plate model that has an external load and is reinforced by an infinite number of equally spaced ribs. The Heaviside step function is used to load the rib forces onto the plate. A Fourier series then replaces the Heaviside step function, and it is shown that, with this substitution, the equation decouples using an orthogonalization procedure. The resulting system can be represented by an infinite set of algebraic equations. These equations are truncated, and a solution to the plate displacement is found. An example problem is formulated and the results are compared to finite element theory to ensure that the proper analytical solution has been obtained. The results are discussed.

2. SYSTEM MODEL AND ANALYTICAL SOLUTION

The equation governing the motion of a plate with an applied external force and an infinite set of finite width stiffeners can be derived through a force balance along the length of the plate. This differential equation written in the spatial-time domain is

$$D\frac{\partial^4 w(x,t)}{\partial x^4}+\rho h\frac{\partial^2 w(x,t)}{\partial t^2}=-f(x,t)-\frac{K}{b}\sum_{n=-\infty}^{n=+\infty}w(x,t)\mathrm{H}(x-nL)+\frac{K}{b}\sum_{n=-\infty}^{n=+\infty}w(x,t)\mathrm{H}(x-b-nL)\,, \tag{1}$$

where $w(x,t)$ is the transverse displacement of the plate in the y-direction (m), x is the spatial location on the plate (m), t is time (s), ρ is the density of the plate (kg m^{-3}), h is the height of the plate (m), $f(x,t)$ is the applied external load on the plate (N m^{-2}), K is the stiffness of each rib per unit length (N m^{-2}), L is the distance between the left edges of adjacent ribs (m), b is the width of each rib (m), H is the Heaviside step function* (dimensionless), and D is the flexural rigidity of the plate, which is given by

$$D=\frac{Eh^3}{12(1-\upsilon^2)}\,, \tag{2}$$

where υ is Poisson's ratio of the plate (dimensionless) and E is Young's modulus of the plate (N m^{-2}). In equation (1), the first term with the Heaviside function corresponds to the left edge of the ribs and the second term corresponds to the right edge of the ribs. A diagram of the system subjected to a continuous spatial force is shown in figure 1. The forcing function and the response are harmonic in time (i.e., $f(x,t)=F(x)\exp(-\mathrm{i}\omega t)$ and $w(x,t)=W(x)\exp(-\mathrm{i}\omega t)$); thus, equation (1) can be written in the spatial-frequency domain with an applied force at a definite wavenumber as

$$D\frac{\mathrm{d}^4 W(x)}{\mathrm{d}x^4}-\rho h\omega^2 W(x)=-F_0\exp(\mathrm{i}kx)-\frac{K}{b}\sum_{n=-\infty}^{n=+\infty}W(x)\mathrm{H}(x-nL)+\frac{K}{b}\sum_{n=-\infty}^{n=+\infty}W(x)\mathrm{H}(x-b-nL)\;,$$

---

*The Heaviside step function, H, also called the unit step function, is a discontinuous function whose value is zero for negative argument and one for positive argument. It seldom matters what value is used for H(0), since H is mostly used as a distribution. The function is used in the mathematics of control theory and signal processing to represent a signal that switches on at a specified time and stays switched on indefinitely.

(3)

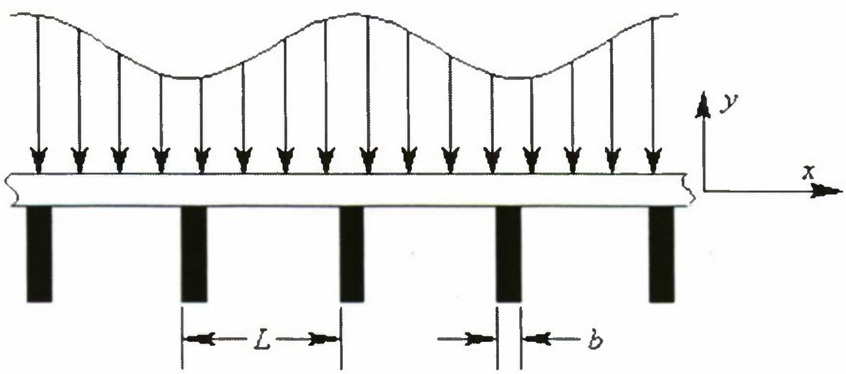

Figure 1. Schematic of Reinforced Plate with an Applied Spatial Force

where F_0 is the magnitude of the applied force (N m^{-2}) and k is the wavenumber of the applied load (rad m^{-1}). Because of the spatial periodicity of the system, the magnitude of the response can be expressed as[9]

$$W(x) = \sum_{m=-\infty}^{m=+\infty} W_m \exp(\mathrm{i}k_m x) \ , \tag{4}$$

where

$$k_m = k + \frac{2\pi m}{L} \ , \tag{5}$$

so that $k_0 \equiv k$, and the W_m's are the unknown coefficients whose solutions are sought. Substituting equation (4) into equation (3) and evaluating the derivatives yields

$$\sum_{m=-\infty}^{m=+\infty} (Dk_m^4 - \rho h\omega^2) W_m \exp(\mathrm{i}k_m x)$$

$$= -F_0 \exp(\mathrm{i}kx) + \frac{K}{b} \sum_{n=-\infty}^{n=+\infty} \left[\sum_{m=-\infty}^{m=+\infty} W_m \exp(\mathrm{i}k_m x) \right] [\mathrm{H}(x-b-nL) - \mathrm{H}(x-nL)] \ , \tag{6}$$

which is an algebraic summation problem free of differentials.

The presence of the Heaviside step functions in the third term complicates the form of equation (6). However, if the Heaviside functions are expressed as a Fourier series, not only are the difficulties associated with these discontinuous functions eliminated, but also each term of the equation becomes an exponential function of x. The Fourier series of the Heaviside functions is written as

$$\sum_{n=-\infty}^{n=+\infty}[\mathrm{H}(x-nL)-\mathrm{H}(x-b-nL)] = \sum_{n=-\infty}^{n=+\infty} c_n \exp(\mathrm{i}2\pi nx/L)\,, \tag{7}$$

where

$$c_n = \begin{cases} \dfrac{1-\exp(-\mathrm{i}2\pi nb/L)}{\mathrm{i}2\pi n} & n \neq 0 \\ \dfrac{b}{L} & n = 0 \end{cases} . \tag{8}$$

Inserting this Fourier series identity into the third term of equation (6) and extracting functions independent of n and m from the double summation yields the term

$$\begin{aligned} &\frac{K}{b}\sum_{n=-\infty}^{n=+\infty}\left[\sum_{m=-\infty}^{m=+\infty} W_m \exp(\mathrm{i}k_m x)\right][\mathrm{H}(x-b-nL)-\mathrm{H}(x-nL)] \\ &= -\frac{K}{b}\exp(\mathrm{i}kx)\sum_{n=-\infty}^{n=+\infty}\left[\sum_{m=-\infty}^{m=+\infty} W_m \exp(\mathrm{i}2\pi mx/L)\right] c_n \exp(\mathrm{i}2\pi nx/L)\,. \end{aligned} \tag{9}$$

Furthermore, the double summation in equation (9) can be rewritten as

$$\begin{aligned} &-\frac{K}{b}\exp(\mathrm{i}kx)\sum_{n=-\infty}^{n=+\infty}\left[\sum_{m=-\infty}^{m=+\infty} W_m \exp(\mathrm{i}2\pi mx/L)\right] c_n \exp(\mathrm{i}2\pi nx/L) \\ &= -\frac{K}{b}\sum_{n=-\infty}^{n=+\infty}\sum_{m=-\infty}^{m=+\infty} W_n c_{m-n} \exp(\mathrm{i}k_m x)\,. \end{aligned} \tag{10}$$

The proof of the double summation identity in equation (10) is presented in the appendix. Inserting the second part of equation (10) into equation (6) yields

$$\sum_{m=-\infty}^{m=+\infty}(Dk_m^4 - \rho h\omega^2)W_m \exp(\mathrm{i}k_m x) = -F_0 \exp(\mathrm{i}kx) - \frac{K}{b}\sum_{n=-\infty}^{n=+\infty}\ \sum_{m=-\infty}^{m=+\infty} W_n\, c_{m-n} \exp(\mathrm{i}k_m x) \ , \tag{11}$$

which is the algebraic equation that models the stiffened plate subjected to an applied force.

The solution to the unknown W_m coefficients is now found by an orthogonal expansion of equation (11). Specifically, equation (11) is multiplied by the exponential $\exp(-\mathrm{i}k_p x)$ and the resulting expression is integrated over $[0,L]$. Because the exponential functions $\exp(-\mathrm{i}k_p x)$ and $\exp(\mathrm{i}k_m x)$ are orthogonal on this interval when $m \neq p$, equation (11) decouples into an infinite number of individual p-indexed equations, each one expressed as

$$(Dk_p^4 - \rho h\omega^2)W_p + \frac{K}{b}\sum_{n=-\infty}^{n=+\infty} c_{p-n} W_n = \begin{cases} -F_0 & p = 0 \\ 0 & p \neq 0 \end{cases} . \tag{12}$$

Writing out all the equations from equation (12) for $-\infty \leq p \leq \infty$ and placing them into matrix form yields

$$\begin{bmatrix} \ddots & & \vdots & & ⋰ \\ & A_{-1} & 0 & 0 & \\ \cdots & 0 & A_0 & 0 & \cdots \\ & 0 & 0 & A_1 & \\ ⋰ & & \vdots & & \ddots \end{bmatrix} \begin{Bmatrix} \vdots \\ W_{-1} \\ W_0 \\ W_1 \\ \vdots \end{Bmatrix} + \frac{K}{b} \begin{bmatrix} \ddots & & \vdots & & ⋰ \\ & c_0 & c_{-1} & c_{-2} & \\ \cdots & c_1 & c_0 & c_{-1} & \cdots \\ & c_2 & c_1 & c_0 & \\ ⋰ & & \vdots & & \ddots \end{bmatrix} \begin{Bmatrix} \vdots \\ W_{-1} \\ W_0 \\ W_1 \\ \vdots \end{Bmatrix} = \begin{Bmatrix} \vdots \\ 0 \\ -F_0 \\ 0 \\ \vdots \end{Bmatrix} , \tag{13}$$

where

$$A_p = Dk_p^4 - \rho h\omega^2 \ . \tag{14}$$

The first term of equation (13) represents the dynamics of the plate, the second term represents the dynamics of the ribs, and the third term represents the external load on the structure. Note that, mathematically, the effect of the ribs is the constant (K/b) multiplied by a matrix that contains a permutation of the Fourier coefficients of the Heaviside step function. The unknown coefficients W_m in equation (13) can be solved by truncating the matrices and vectors to a finite number of terms and analytically evaluating the equation

$$\begin{Bmatrix} \vdots \\ W_{-1} \\ W_0 \\ W_1 \\ \vdots \end{Bmatrix} = \left[\begin{bmatrix} \ddots & & \vdots & & \iddots \\ & A_{-1} & 0 & 0 & \\ \cdots & 0 & A_0 & 0 & \cdots \\ & 0 & 0 & A_1 & \\ \iddots & & \vdots & & \ddots \end{bmatrix} + \frac{K}{b} \begin{bmatrix} \ddots & & \vdots & & \iddots \\ & c_0 & c_{-1} & c_{-2} & \\ \cdots & c_1 & c_0 & c_{-1} & \cdots \\ & c_2 & c_1 & c_0 & \\ \iddots & & \vdots & & \ddots \end{bmatrix} \right]^{-1} \begin{Bmatrix} \vdots \\ 0 \\ -F_0 \\ 0 \\ \vdots \end{Bmatrix} . \qquad (15)$$

3. NUMERICAL EXAMPLE

A numerical example is now formulated and discussed. The problem consists of a thin plate that has a Young's modulus of 3×10^8 N m^{-2}, density of 1200 kg m^{-3}, Poisson's ratio of 0.45, and a height of 0.01 m. Each rib has a stiffness per unit length of 1×10^8 N m^{-2} and a width of 0.2 m. The center-to-center spacing of adjacent ribs is 1 m. The applied load has a magnitude of 1 N m^{-2} and a wavenumber of 0 rad m^{-1}. These parameters are input into equation (15) to determine the coefficients that are, in turn, entered into equation (4) to calculate the displacement of the plate. Figure 2 is a plot of the magnitude of plate displacement in the decibel scale versus the spatial location at a frequency of 50 Hz. The solid line is the solution derived in this report, while the dots are a verification solution determined using the finite element method. The dashed line on the plot indicates the spatial position of the right edge of the rib that corresponds to $n = 0$ in equation (1). This specific problem converged using 31 terms ($-15 \le p \le 15$), although it is noted that convergence is a function of the specific parameters used in the model.

Several features in figure 2 are now noted. First, the plate/rib system is softer between the ribs and, thus, has a greater displacement at these spatial locations. As a result, if the stiffness of the ribs approaches 0, the solution approaches that of an unreinforced plate. Second, even

though the ribs are fairly stiff in this example, they still exhibit dynamic behavior across their spatial extent. As evidence, examine figure 2 in the area from $x = 0$ to $x = b = 0.2$. Each rib has almost 40 dB of motion in this relatively small space, although the displacement is significantly lower than the plate displacement between the ribs. Third, although not explicitly shown in figure 2, the solution is periodic on $x \in [0, L]$. Sectional aperiodocity can be added to the model using previously developed techniques.[5]

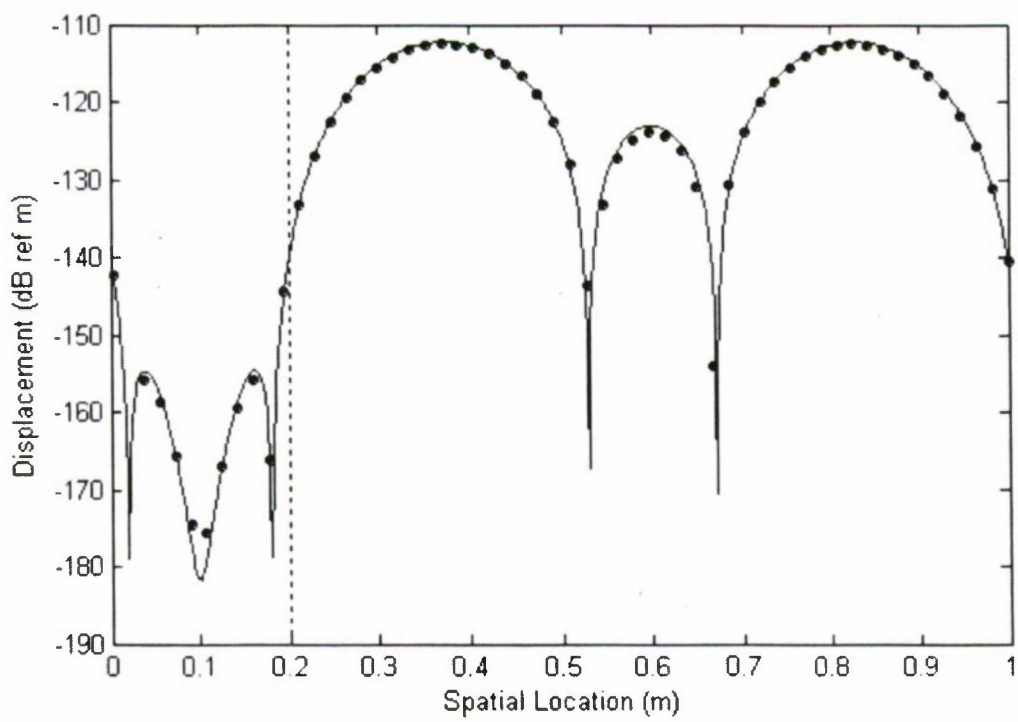

Figure 2. Magnitude of Plate Displacement Versus Spatial Location Showing the Analytical Solution (—) and the Finite Element Solution (•)

(The dashed line is the location of the right edge of the $n = 0$ rib.)

4. CONCLUSIONS

A model of a reinforced thin plate with ribs of finite thickness was derived and the solution was obtained using an orthogonalization procedure. It was shown that the effect of the ribs is, mathematically, a constant multiplied by a matrix that contains a permutation of the Fourier coefficients of the Heaviside step function. A numerical example was presented and compared to a finite element analysis. This new analytical solution was in almost exact agreement with the finite element solution. The results of the model and the associated dynamics were briefly discussed.

REFERENCES

1. B. R. Mace, "Periodically Stiffened Fluid-Loaded Plates, I: Response to Convected Harmonic Pressure and Free Wave Propagation," *Journal of Sound and Vibration*, vol. 73, no. 4, 1980, pp. 473–486.

2. B. R. Mace, "Periodically Stiffened Fluid-Loaded Plates, II: Response to Line and Point Forces," *Journal of Sound and Vibration*, vol. 73, no.4, 1980, pp. 487–504.

3. P. R. Stepanishen, "The Acoustic Transmission and Scattering Characteristics of a Plate with Line Impedance Discontinuities," *Journal of Sound and Vibration*, vol. 58, no. 2, 1978, pp. 257–272.

4. G. P. Eatwell and D. Butler, "The Response of a Fluid-Loaded, Beam-Stiffened Plate," *Journal of Sound and Vibration*, vol. 84, no.3, 1982, pp. 371–388.

5. B. A. Cray, "Acoustic Radiation from Periodic and Sectionally Aperiodic Rib-Stiffened Plates," *Journal of the Acoustical Society of America*, vol. 95, no. 1, 1994, pp. 256–264.

6. A. J. Hull, "Elastic Response of an Acoustic Coating on a Rib-Stiffened Plate," *Journal of Sound and Vibration*, vol. 329, no. 20, 2010, pp. 4192–4211.

7. B. L. Woolley, "Acoustic Scattering from a Submerged Plate. I. One Reinforcing Rib," *Journal of the Acoustical Society of America*, vol. 67, no. 5, 1980, pp. 1642–1653.

8. B. L. Woolley, "Acoustic Scattering from a Submerged Plate. II. Finite Number of Reinforcing Ribs," *Journal of the Acoustical Society of America*, vol. 67, no. 5, 1980, pp. 1654–1658.

9. D. J. Mead and K. K. Pujara, "Space-Harmonic Analysis of Periodically Supported Beams: Response to Convected Random Loading," *Journal of Sound and Vibration*, vol. 14, no. 4, 1971, pp. 525–541.

APPENDIX
DOUBLE SUMMATION IDENTITY

The proof of the double summation identity given in equation (10) is presented in this appendix. The identity is critical in transforming the problem into a tractable algebraic equation. The first term in equation (10) without the (K/b) coefficient is

$$\exp(ikx)\sum_{n=-\infty}^{n=+\infty}\left[\sum_{m=-\infty}^{m=+\infty}W_m \exp(i2\pi mx/L)\right]c_n \exp(i2\pi nx/L) \ . \qquad \text{(A-1)}$$

To simplify the notation of the following equations, the formula

$$u_m = \exp(i2\pi mx/L) \qquad \text{(A-2)}$$

is used. This term has the property

$$u_{m+n} = u_m u_n \ . \qquad \text{(A-3)}$$

Expanding the m indexed series results in

$$\exp(ikx)\sum_{n=-\infty}^{n=+\infty}[\cdots + W_{-2}u_{-2} + W_{-1}u_{-1} + W_0u_0 + W_1u_1 + W_2u_2 + \cdots]c_n u_n \ . \qquad \text{(A-4)}$$

Next, expanding the n indexed series yields

$$\begin{aligned}\exp(ikx)\big[\cdots &+ (\cdots + W_{-2}u_{-2} + W_{-1}u_{-1} + W_0u_0 + W_1u_1 + W_2u_2 + \cdots)c_{-2}u_{-2}\\ &+ (\cdots + W_{-2}u_{-2} + W_{-1}u_{-1} + W_0u_0 + W_1u_1 + W_2u_2 + \cdots)c_{-1}u_{-1}\\ &+ (\cdots + W_{-2}u_{-2} + W_{-1}u_{-1} + W_0u_0 + W_1u_1 + W_2u_2 + \cdots)c_0u_0\\ &+ (\cdots + W_{-2}u_{-2} + W_{-1}u_{-1} + W_0u_0 + W_1u_1 + W_2u_2 + \cdots)c_1u_1\\ &+ (\cdots + W_{-2}u_{-2} + W_{-1}u_{-1} + W_0u_0 + W_1u_1 + W_2u_2 + \cdots)c_2u_2 + \cdots\big] \ , \end{aligned} \qquad \text{(A-5)}$$

and multiplying through gives

$$\exp(\mathrm{i}kx)\big[\cdots + (\cdots + W_{-2}c_{-2}u_{-4} + W_{-1}c_{-2}u_{-3} + W_0c_{-2}u_{-2} + W_1c_{-2}u_{-1} + W_2c_{-2}u_0 + \cdots)$$
$$+ (\cdots + W_{-2}c_{-1}u_{-3} + W_{-1}c_{-1}u_{-2} + W_0c_{-1}u_{-1} + W_1c_{-1}u_0 + W_2c_{-1}u_1 + \cdots)$$
$$+ (\cdots + W_{-2}c_0u_{-2} + W_{-1}c_0u_{-1} + W_0c_0u_0 + W_1c_0u_1 + W_2c_0u_2 + \cdots)$$
$$+ (\cdots + W_{-2}c_1u_{-1} + W_{-1}c_1u_0 + W_0c_1u_1 + W_1c_1u_2 + W_2c_1u_3 + \cdots)$$
$$+ (\cdots + W_{-2}c_2u_0 + W_{-1}c_2u_1 + W_0c_2u_2 + W_1c_2u_3 + W_2c_2u_4 + \cdots) + \cdots\big] . \qquad \text{(A-6)}$$

Regrouping equation (A-6) on specific values of W_n yields the equation

$$\exp(\mathrm{i}kx)\left\{\cdots + \sum_{m=-\infty}^{m=+\infty} W_{-2}[c_{m-(-2)}]u_m + \sum_{m=-\infty}^{m=+\infty} W_{-1}[c_{m-(-1)}]u_m + \sum_{m=-\infty}^{m=+\infty} W_0[c_{m-0}]u_m\right.$$
$$\left. + \sum_{m=-\infty}^{m=+\infty} W_1[c_{m-1}]u_m + \sum_{m=-\infty}^{m=+\infty} W_2[c_{m-2}]u_m + \cdots\right\} , \qquad \text{(A-7)}$$

which can be rewritten as

$$\exp(\mathrm{i}kx)\sum_{n=-\infty}^{n=+\infty}\sum_{m=-\infty}^{m=+\infty} u_m W_n c_{m-n} = \sum_{n=-\infty}^{n=+\infty}\sum_{m=-\infty}^{m=+\infty} W_n c_{m-n}\exp(\mathrm{i}k_m x) . \qquad \text{(A-8)}$$

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